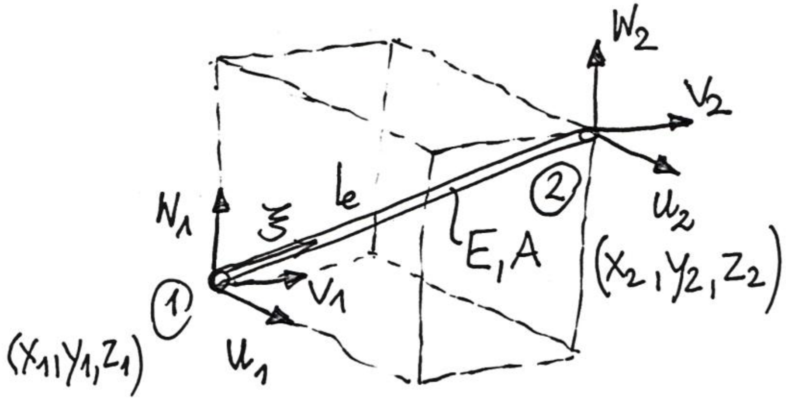
Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

Lecture 8A. 3D Truss Bar Finite Element

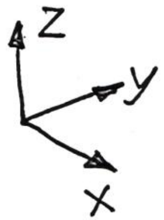
04.2025

3D Truss Bar Finite Element



$$\{q_g\}_e_{6\times 1} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}_e$$

Global vector of nodal parameters

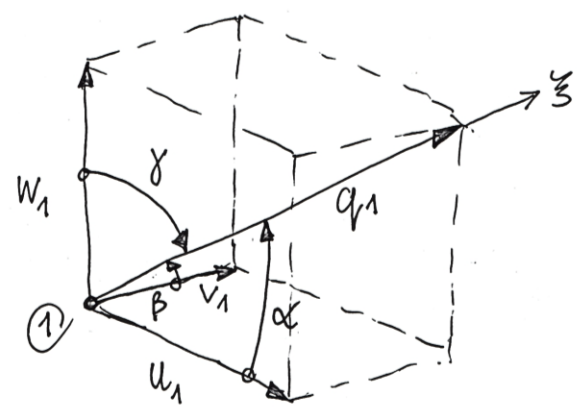


$$l_e = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Local vector of nodal parameters

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e$$

Displacement of node 1

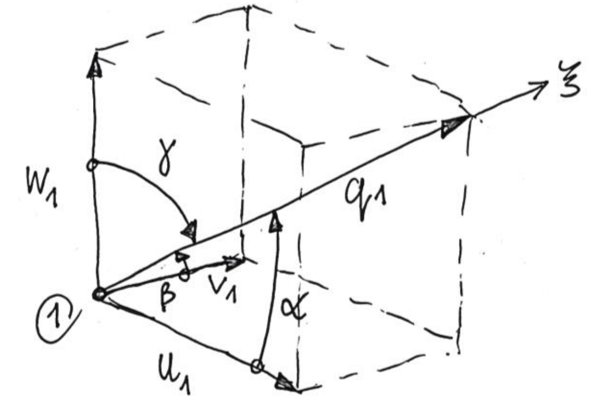


$$\cos\alpha = \frac{x_2-x_1}{l_e}$$

$$\cos\beta = \frac{y_2-y_1}{l_e}$$

$$\cos\gamma = \frac{z_2-z_1}{l_e}$$

Displacement of node 1



$$u_1 = q_1 \cos\alpha$$
$$v_1 = q_1 \cos\beta$$
$$w_1 = q_1 \cos\gamma$$

$$u_1 \cos\alpha = q_1 \cos^2\alpha$$
$$v_1 \cos\beta = q_1 \cos^2\beta$$
$$w_1 \cos\gamma = q_1 \cos^2\gamma$$

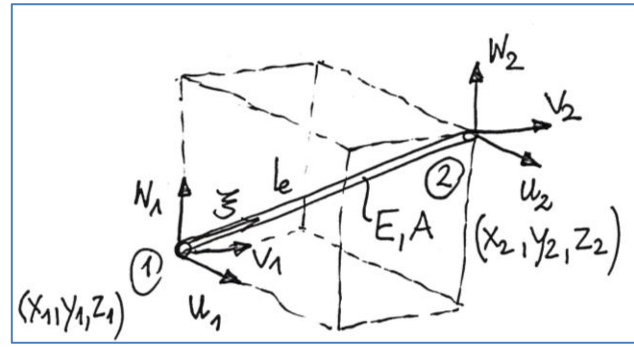
$$u_1 \underbrace{\cos\alpha}_{a} + v_1 \underbrace{\cos\beta}_{b} + w_1 \underbrace{\cos\gamma}_{c} = q_1$$

Displacements of node 1 and 2

$$q_1 = a \cdot u_1 + b \cdot v_1 + c \cdot w_1 + 0 \cdot u_2 + 0 \cdot v_2 + 0 \cdot w_2$$
$$q_2 = 0 \cdot u_1 + 0 \cdot v_1 + 0 \cdot w_1 + a \cdot u_2 + b \cdot v_2 + c \cdot w_2$$

Vector of nodal displacements of a truss element

$$q_1 = a \cdot u_1 + b \cdot v_1 + c \cdot w_1 + 0 \cdot u_2 + 0 \cdot v_2 + 0 \cdot w_2$$
$$q_2 = 0 \cdot u_1 + 0 \cdot v_1 + 0 \cdot w_1 + a \cdot u_2 + b \cdot v_2 + c \cdot w_2$$



$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \underbrace{\begin{bmatrix} a & b & c & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & c \end{bmatrix}}_{} \cdot \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}_e$$

$$\underset{2\times1}{\{q\}_e} = \underset{2\times6}{[T_t]_e} \cdot \underset{6\times1}{\{q_g\}_e}$$

$$\underset{1\times2}{\lfloor q \rfloor_e} = \underset{1\times6}{\lfloor q_g \rfloor} \cdot \underset{6\times2}{[T_t]_e^T} \quad ; \quad [T_t]_e^T = \begin{bmatrix} a & 0 \\ b & 0 \\ c & 0 \\ 0 & a \\ 0 & b \\ 0 & c \end{bmatrix}$$

Transformation matrix

Elastic strain energy of a truss element:

$$U_e = \frac{1}{2} \underset{1\times 2}{\lfloor q \rfloor_e} \cdot \underset{2\times 2}{[K]_e} \cdot \underset{2\times 1}{\{q\}_e} = \frac{1}{2} \underset{1\times 6}{\lfloor q_g \rfloor_e} \cdot \underset{6\times 2}{[T_t]_e^T} \underset{2\times 2}{[K]_e} \underset{2\times 6}{[T_t]_e} \cdot \underset{6\times 1}{\{q_g\}_e} =$$

$$= \frac{1}{2} \underset{1\times 6}{\lfloor q_g \rfloor_e} \underset{6\times 6}{[K_g]_e} \cdot \underset{6\times 1}{\{q_g\}_e}$$

, where:

$$\underset{2\times 2}{[k]_e} = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global stiffness matrix of a truss element:

$$\underset{6\times 6}{[k_g]_e} = \frac{EA}{l_e} \begin{bmatrix} a^2 & ab & ac & -a^2 & -ab & -ac \\ ab & b^2 & bc & -ab & -b^2 & -bc \\ ac & bc & c^2 & -ac & -bc & -c^2 \\ -a^2 & -ab & -ac & a^2 & ab & ac \\ -ab & -b^2 & -bc & ab & b^2 & bc \\ -ac & -bc & -c^2 & ac & bc & c^2 \end{bmatrix}$$

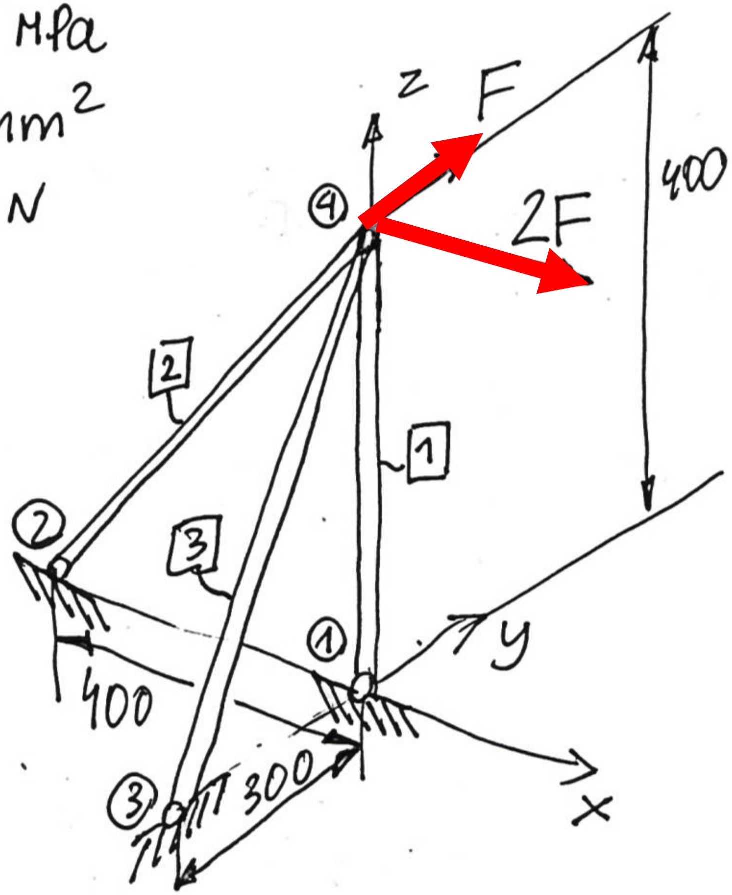
Example Build a finite element model of a 3D truss.

Find nodal displacements, stress, internal forces and reactions

$E = 2 \cdot 10^5$ MPa

$A = 100 \text{ mm}^2$

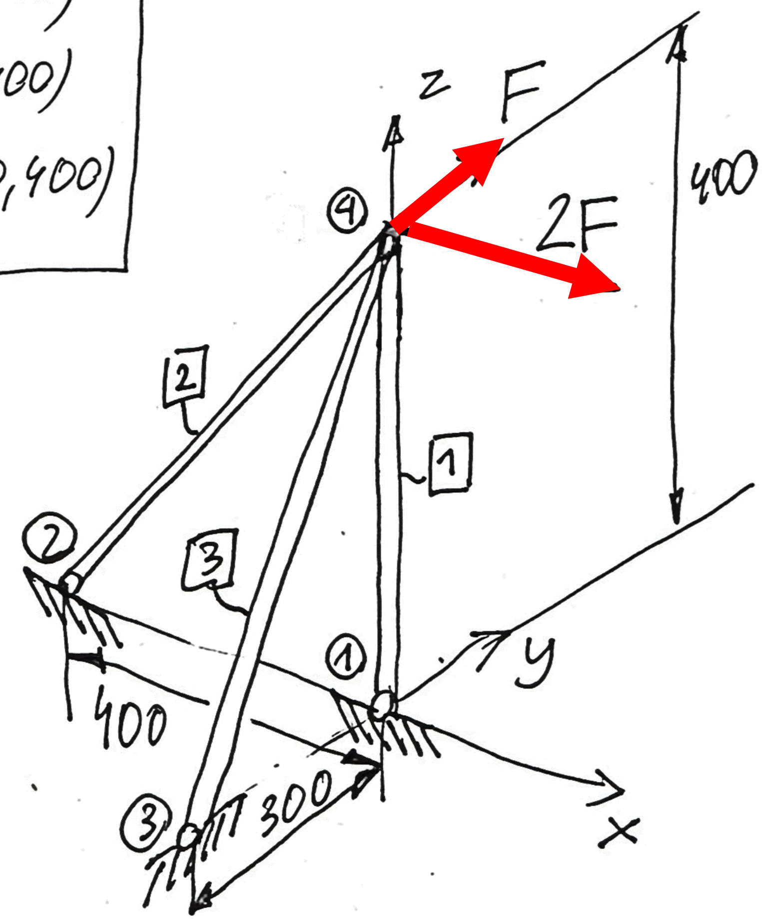
$F = 1500$ N



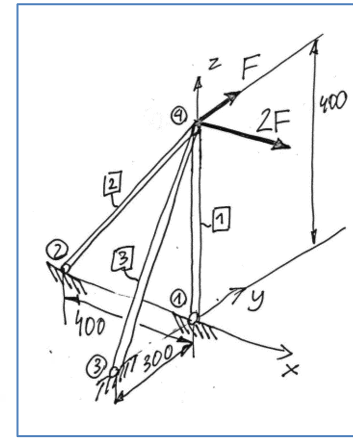
$$\{q\}_{12\times1} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

FE Model

ELEMENT	NODES
1	1 (0, 0, 0) → 4 (0, 0, 400)
2	2 (−400, 0, 0) → 4 (0, 0, 400)
3	3 (0, −300, 0) → 4 (0, 0, 400)



Stiffness matrix of element 1

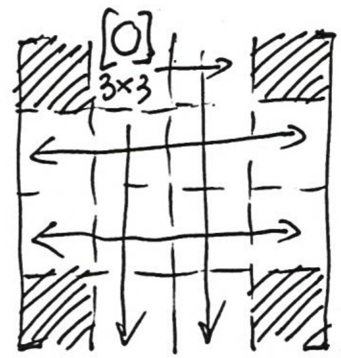


ELEMENT $\boxed{1}$; $l_1 = 400\,mm$

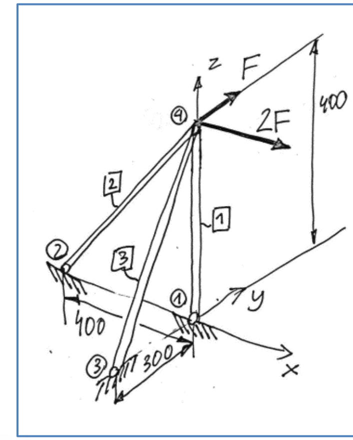
$$a_1 = \frac{0-0}{l_1} = 0 \quad ; \quad b_1 = \frac{0-0}{l_1} = 0, \quad c_1 = \frac{400-0}{400} = 1$$

$$\underset{2\times 6}{[T_t]_1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{6\times 6}{[k_g]_1} = \frac{EA}{l_1}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underset{12\times 12}{[k_g]_1^*} =$$



Stiffness matrix of element 2



ELEMENT $\boxed{2}$

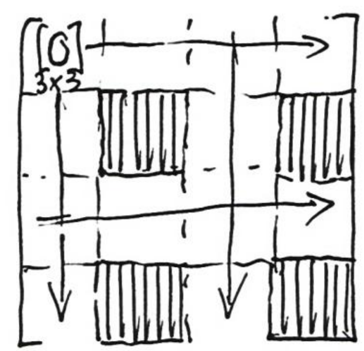
$$l_2 = \sqrt{(0-(-400))^2 + (0-0)^2 + (400-0)^2} = 400\sqrt{2}\ \text{mm}$$

$$a_2 = \frac{0-(-400)}{400\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad b_2 = \frac{0-0}{400\sqrt{2}} = 0, \quad c_2 = \frac{400-0}{400\sqrt{2}} = \frac{\sqrt{2}}{2}$$

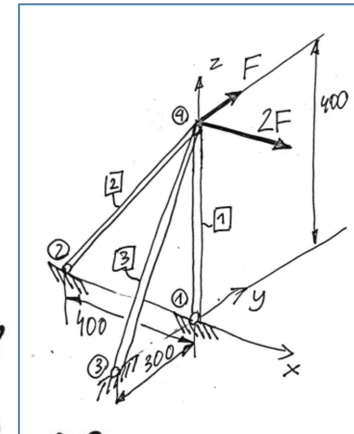
$$\underset{2\times 6}{[T_t]_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\underset{6\times 6}{[k_g]_2} = \frac{EA}{l_2} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\underset{12\times 12}{[k_g]_2^*} =$$



Stiffness matrix of element 3



ELEMENT $\boxed{3}$

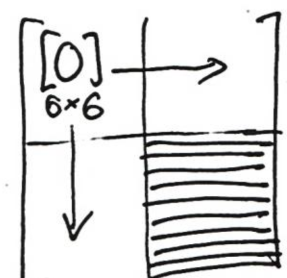
$$l_3 = \sqrt{(0-0)^2 + (0-(-300))^2 + (400-0)^2} = 500\,mm$$

$$a_3 = \frac{0-0}{500} = 0 \,, \quad b_3 = \frac{0-(-300)}{500} = \frac{3}{5} = 0.6 \,, \quad c_3 = \frac{400-0}{500} = 0.8$$

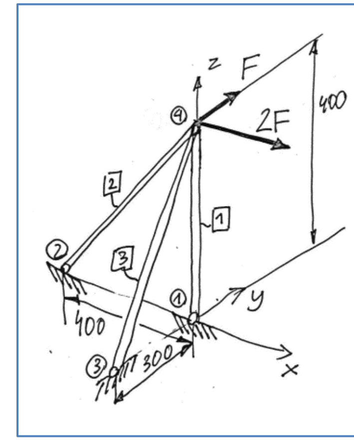
$$\underset{2\times 6}{[T_t]_3} = \begin{bmatrix} 0 & 0.6 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.8 \end{bmatrix}$$

$$\underset{6\times 6}{[k_g]_3} = \frac{EA}{l_3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.36 & 0.48 & 0 & -0.36 & -0.48 \\ 0 & 0.48 & 0.64 & 0 & -0.48 & -0.64 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.36 & -0.48 & 0 & 0.36 & 0.48 \\ 0 & -0.48 & -0.64 & 0 & 0.48 & 0.64 \end{bmatrix}$$

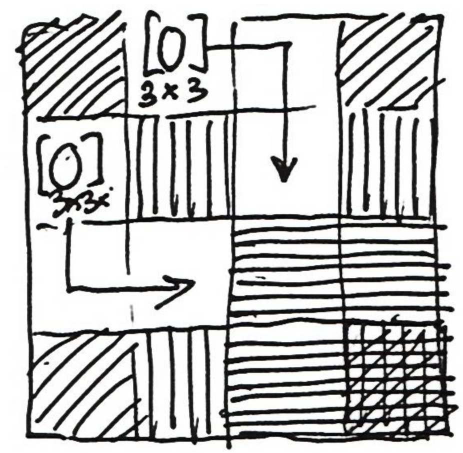
$$\underset{12\times 12}{[k_g]_3^*} =$$



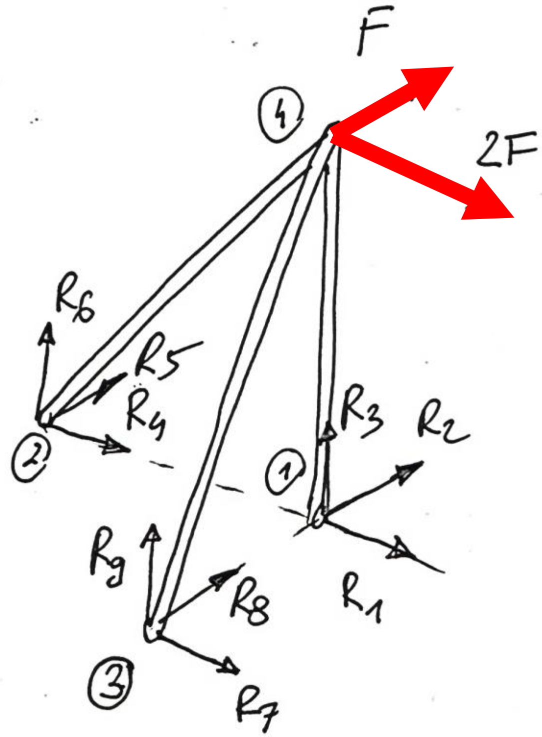
Global stiffness matrix of the truss model



$$\underset{12\times 12}{[K]} = [k_g]_1^* + [k_g]_2^* + [k_g]_3^* =$$

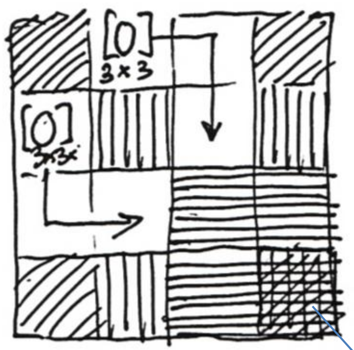


Global Load Vector:



$$\underset{12\times 1}{\{F\}} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \\ 2F \\ F \\ 0 \end{Bmatrix}$$

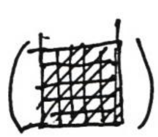
Global stiffness matrix:



Boundary conditions:

$$u_1 = v_1 = w_1 = u_2 = v_2 = w_2 = u_3 = v_3 = w_3 = 0$$

$$N = NDOF - NOF = 12 - 9 = 3$$



$$\underset{3\times 3}{[K]} \cdot \begin{Bmatrix} u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 2F \\ F \\ 0 \end{Bmatrix}$$

System of equations:

$$\underset{3\times 3}{[K]} = EA \begin{bmatrix} \frac{1}{2l_2} & 0 & \frac{1}{2l_2} \\ 0 & \frac{0.36}{l_3} & \frac{0.48}{l_3} \\ \frac{1}{2l_2} & \frac{0.48}{l_3} & \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3}\right) \end{bmatrix}$$

$$\underset{3\times 3}{[K]} \cdot \begin{Bmatrix} u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 2F \\ F \\ 0 \end{Bmatrix}$$

$$\begin{cases} \frac{1}{2l_2} \cdot u_4 + 0 \cdot v_4 + \frac{1}{2l_2} w_4 = \frac{2F}{EA} & \Big| \cdot 2l_2 \\ 0 \cdot u_4 + \frac{0.36}{l_3} v_4 + \frac{0.48}{l_3} w_4 = \frac{F}{EA} & \Big| \cdot \frac{100 l_3}{36} \\ \frac{1}{2l_2} \cdot u_4 + \frac{0.48}{l_3} v_4 + \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3}\right) w_4 = 0 \end{cases}$$

$$\begin{cases} u_4 + w_4 = \dfrac{4Fl_2}{EA} & \Rightarrow \; u_4 = \dfrac{4Fl_2}{EA} - w_4 \\ v_4 + \dfrac{4}{3}w_4 = \dfrac{100\,Fl_3}{36\,EA} & \Rightarrow \; v_4 = \dfrac{100\,Fl_3}{36\,EA} - \dfrac{4}{3}w_4 \\ \dfrac{1}{2l_2}u_4 + \dfrac{0.48}{l_3}v_4 + \left(\dfrac{1}{l_1} + \dfrac{1}{2l_2} + \dfrac{0.64}{l_3}\right)w_4 = 0 \end{cases}$$

$$\frac{1}{2l_2}\cdot\left(\frac{4Fl_2}{EA} - w_4\right) + \frac{0.48}{l_3}\cdot\left(\frac{100}{36}\frac{Fl_3}{EA} - \frac{4}{3}w_4\right) + \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3}\right)w_4 = 0$$

$$\frac{2F}{EA} + \frac{4F}{3EA} + \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3} - \frac{1}{2l_2} - \frac{0.64}{l_3}\right)w_4 = 0$$

$$w_4 = -\left(\frac{2F}{EA} + \frac{4F}{3EA}\right)l_1 = -\frac{10\,Fl_1}{3\,EA} = -0.1\,mm$$

$$u_4 = \frac{4Fl_2}{EA} + \frac{10\,Fl_1}{3EA} = \frac{12\,Fl_2}{3EA} + \frac{10Fl_1}{3EA} = \frac{(12l_2 + 10l_1)\,F}{3EA} = 0.27\,mm$$

$$v_4 = \frac{100\,Fl_3}{36\,EA} - \frac{4}{3}\cdot\left(-\frac{10}{3}\frac{Fl_1}{EA}\right) = \frac{100Fl_3}{36EA} + \frac{40\,Fl_1}{9\,EA} = \frac{(100l_3 + 160l_1)F}{36EA} = 0.2375\,mm$$

Reactions:

$$[K]_{12\times12}\cdot\{q\}_{12\times1}=\{F\}_{12\times1}$$

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&50000&0&0&0&0&0&0&0&0&-50000\\
0&0&0&17677.67&0&17677.67&0&0&0&-17677.7&0&-17677.7\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&17677.67&0&17677.67&0&0&0&-17677.7&0&-17677.7\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&14400&19200&0&-14400&-19200\\
0&0&0&0&0&0&0&19200&25600&0&-19200&-25600\\
0&0&0&-17677.7&0&-17677.7&0&0&0&17677.67&0&17677.67\\
0&0&0&0&0&0&0&-14400&-19200&0&14400&19200\\
0&0&-50000&-17677.7&0&-17677.7&0&-19200&-25600&17677.67&19200&93277.67
\end{bmatrix}\cdot\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0.269706\\0.2375\\-0.1\end{bmatrix}=\begin{bmatrix}0\\0\\5000\\-3000\\0\\-3000\\0\\-1500\\-2000\\3000\\1500\\0\end{bmatrix}$$

$$R_1=0\,,\quad R_2=0\,,$$

$$R_3=-\frac{EA}{l_1}\cdot w_4=-\frac{EA}{l_1}\cdot\left(-\frac{10}{3}\frac{Fl_1}{EA}\right)=\frac{10}{3}F=5000N$$

$$R_4=-\frac{EA}{2l_2}\cdot u_4-\frac{EA}{2l_2}w_4=-\frac{EAF}{2l_2}\left(\frac{12l_2+10l_1}{3EA}-\frac{10l_1}{3EA}\right)=-2F=-3000N$$

$$R_5=0\,,\quad R_6=-\frac{EA}{2l_2}u_4-\frac{EA}{2l_2}w_4=R_4=-2F=-3000N$$

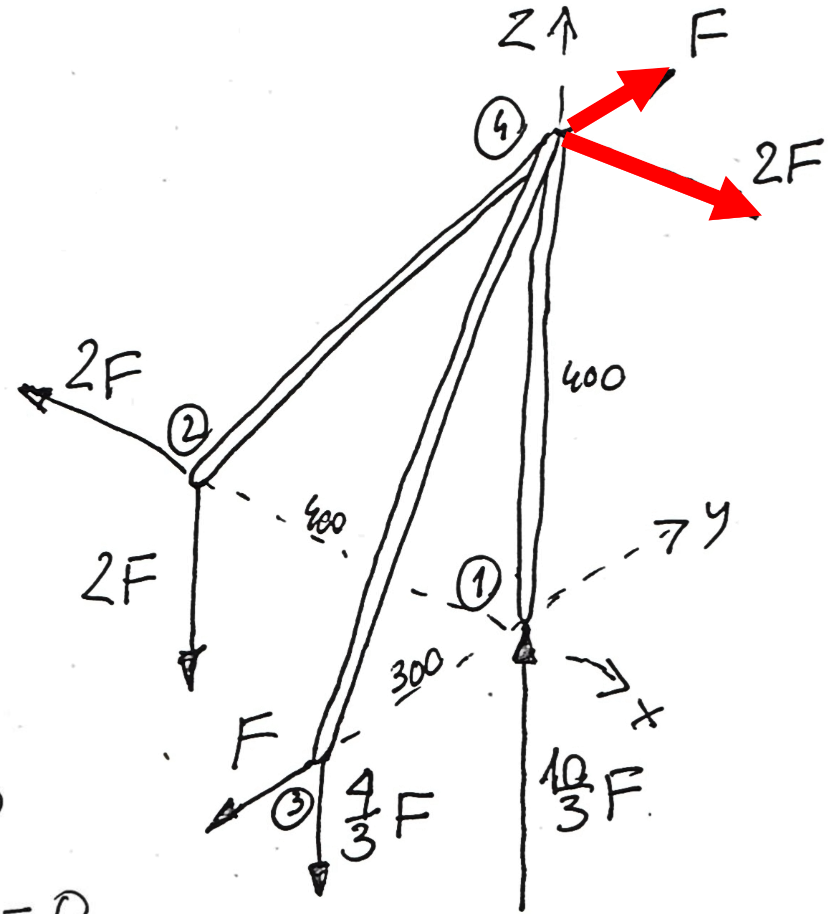
$$R_7=0\,,\quad R_8=-\frac{0.36EA}{l_3}\cdot v_4-\frac{0.48EA}{l_3}w_4=$$

$$=-\frac{0.36EA}{l_3}\cdot\frac{100(l_3+1.6l_1)F}{36EA}+\frac{0.48EA}{l_3}\cdot\frac{10}{3}\frac{Fl_1}{EA}=-F=-1500N$$

$$R_9=\frac{-0.48EA}{l_3}\cdot v_4-\frac{0.64EA}{l_3}w_4=$$

$$=-\frac{0.48EA}{l_3}\cdot\frac{100(l_3+1.6l_1)F}{36EA}+\frac{0.64EA}{l_3}\cdot\frac{10}{3}\frac{Fl_1}{EA}=-\frac{4F}{3}=-2000N$$

Forces in equilibrium:



$$\sum F_x = 0 : \quad -2F + 2F = 0$$

$$\sum F_y = 0 : \quad -F + F = 0$$

$$\sum F_z = 0 : \quad \frac{10}{3}F - 2F - \frac{4}{3}F = 0$$

$$\sum M_x^{(1)} = 0 : \quad \frac{4}{3}F \cdot 300\,\text{mm} - F \cdot 400\,\text{mm} = 0$$

$$\sum M_y^{(1)} = 0 : \quad -2F \cdot 400\,\text{mm} + 2F \cdot 400\,\text{mm} = 0$$

$$\sum M_z^{(2)} = 0 : \quad -F \cdot 400\,\text{mm} + F \cdot 400\,\text{mm} = 0$$

Element 1 solution:

$\boxed{1}$

$$\underset{2\times L}{\{q\}_1} = [T_t]_1 \cdot \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ w_4 \end{Bmatrix}$$

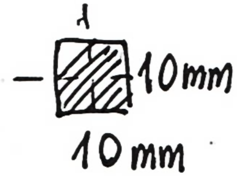
$$\varepsilon_1 = \frac{w_4 - 0}{l_1} = -\frac{10\,F}{3\,EA} = -0.25 \cdot 10^{-3}$$

$$\sigma_1 = E \cdot \varepsilon_1 = -\frac{10}{3}\frac{F}{A} = -50\,MPa \qquad N_1 = \sigma_1 \cdot A = -\frac{10}{3}F = -5000\,N$$

Compression (possible buckling?)

Euler force in element 1:

$$F_{CR} = \frac{\pi^2 E J}{l_1^2}$$



$$J = \frac{(10mm)^4}{12}$$

$$F_{CR} = \frac{\pi^2 E \cdot 10^4 mm^4}{12 \cdot 400^2 mm^2} = \frac{\pi^2 \cdot 2 \cdot 10^5 \cdot 10^4\ Nmm^2}{12 \cdot 16 \cdot 10^4 mm^2} = 10281\,N$$

Safety factor: $n = 2$

$$|N_1| < \frac{P_{kr}}{n}$$

Element 2 solution:

$$\{q\}_2 = \left[\Gamma_t\right]_{2\atop 2\times 6} \cdot \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{\sqrt{2}}{2} \cdot (u_4 + w_4) \end{Bmatrix}$$

$$\varepsilon_2 = \frac{\sqrt{2}}{2l_2}(u_4 + w_4) = \frac{\sqrt{2}}{2l_2} \cdot \left(\frac{(12l_2 + 10l_1)F}{3EA} - \frac{10Fl_1}{3EA} \right) = \frac{2\sqrt{2}F}{EA} = 0.212 \cdot 10^{-3}$$

$$\sigma_2 = E\varepsilon_2 = 42.43\ MPa,$$

$$N_2 = \sigma_2 \cdot A = 4243N$$

Element 3 solution:

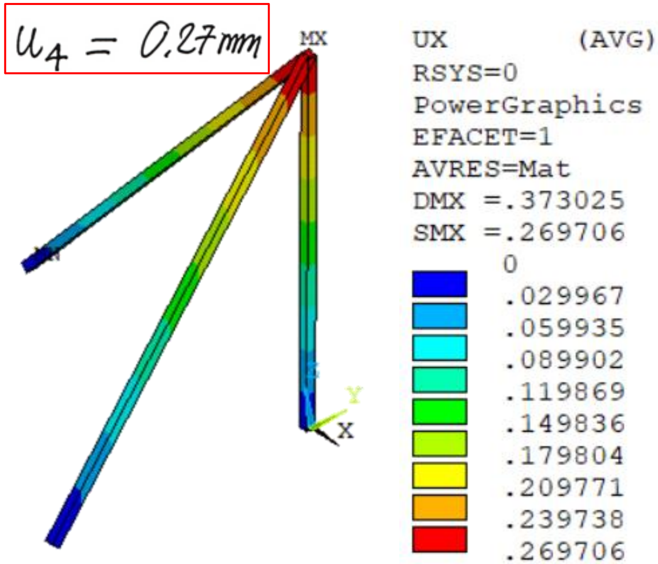
$$\{q\}_3 = [T_t]_3 \cdot \begin{Bmatrix} u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0.6 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.8 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.6 \cdot v_4 + 0.8 \cdot w_4 \end{Bmatrix}$$

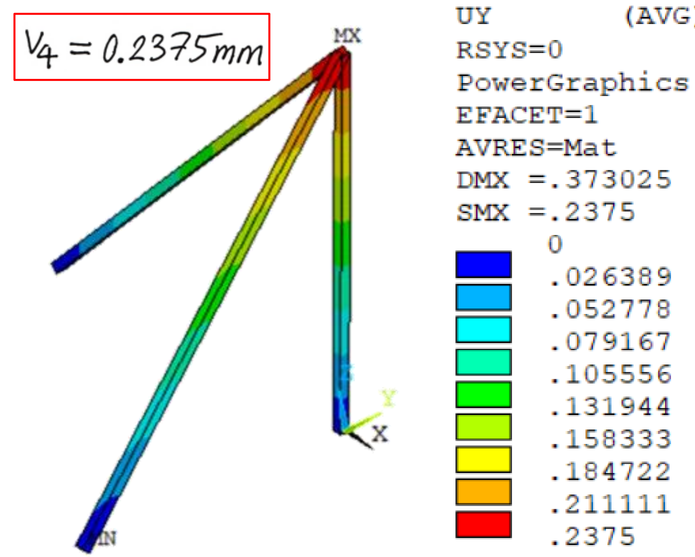
$$\varepsilon_3 = \frac{1}{l_3}\left(0.6 \cdot \frac{(100\, l_3 + 160\, l_1) F}{36\, EA} - 0.8 \cdot \frac{10}{3} \frac{F l_1}{EA}\right) = \frac{5F}{3EA} = 0.125 \cdot 10^{-3}$$

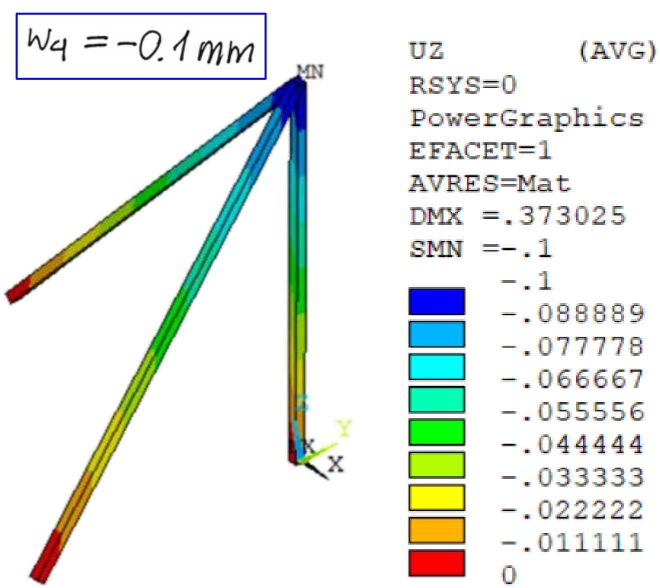
$$\sigma_3 = E \cdot \varepsilon_3 = 25\,MPa$$

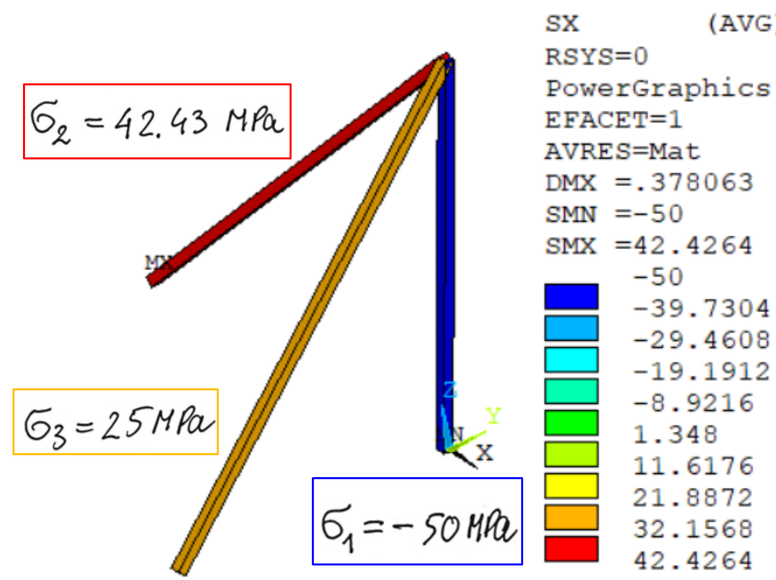
$$N_3 = \sigma_3 \cdot A = 2500\,N$$

Comparison of calculation results with those obtained in the ANSYS program

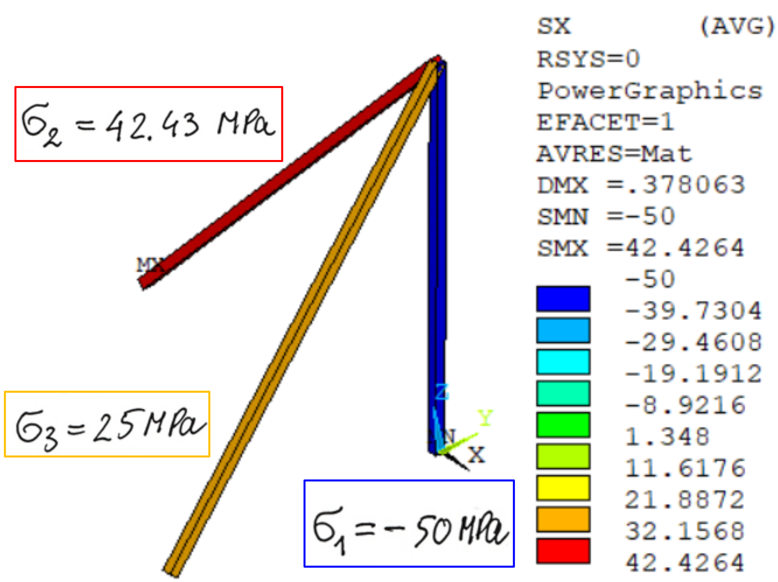


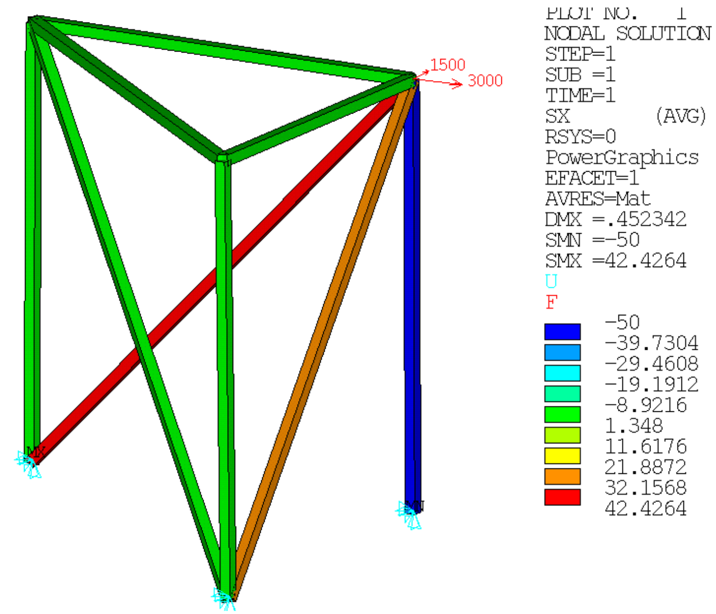


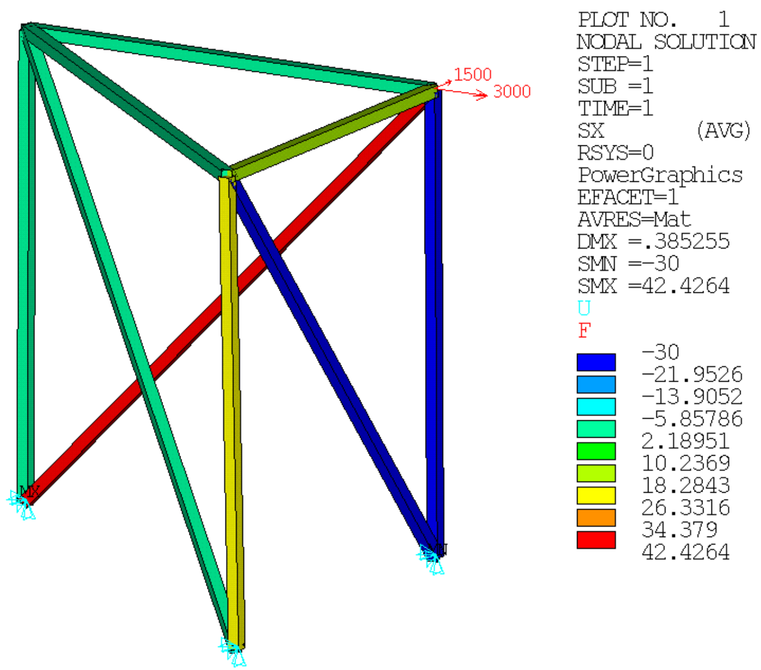


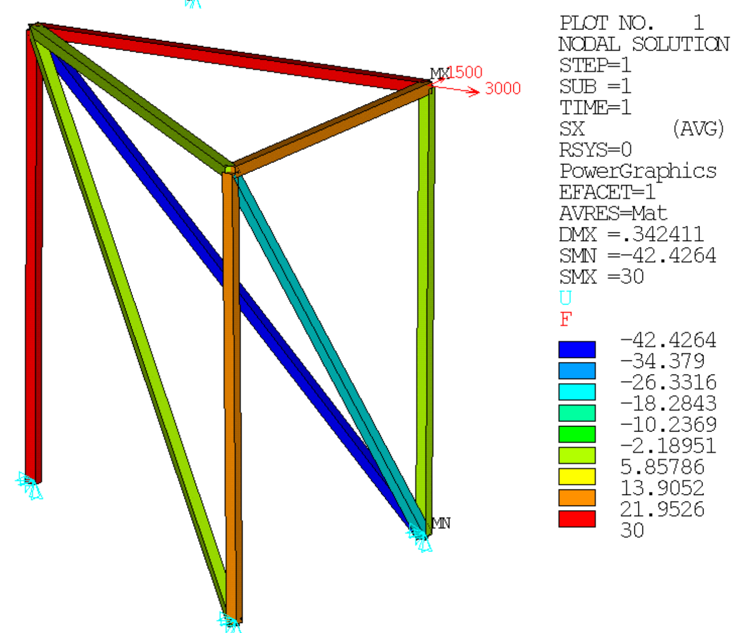


Modification of the model in ANSYS

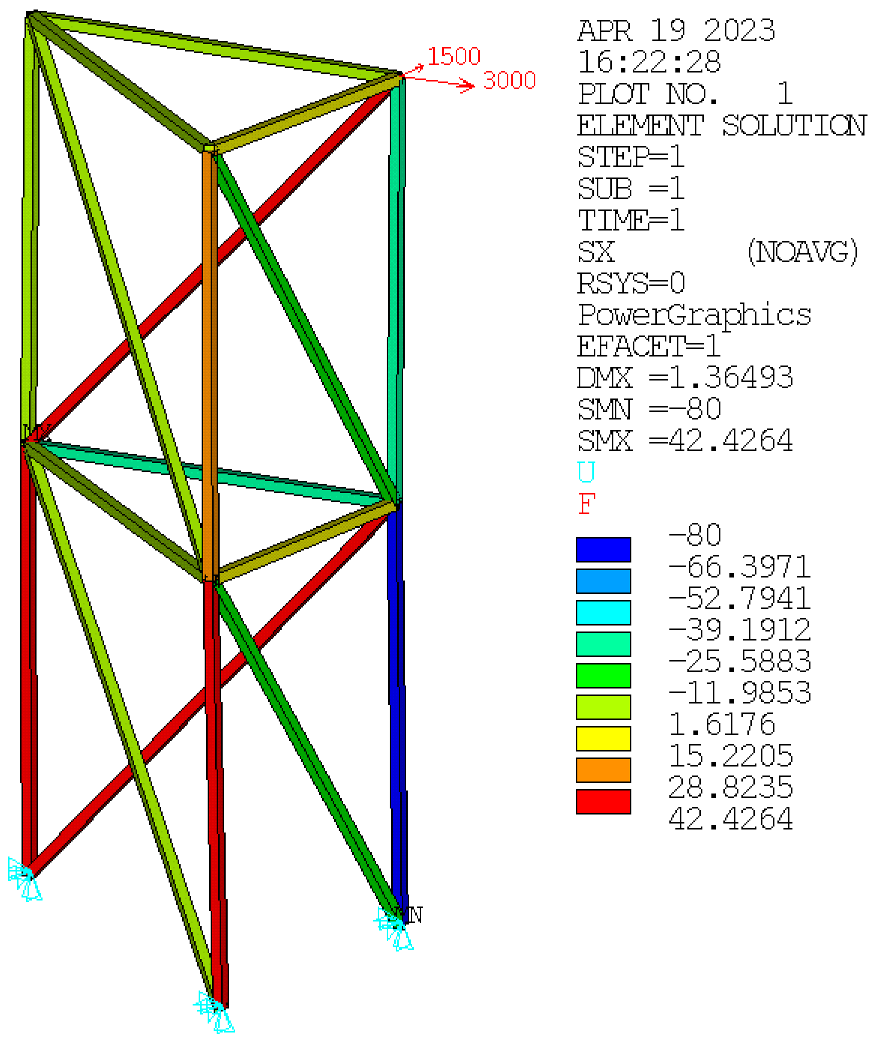


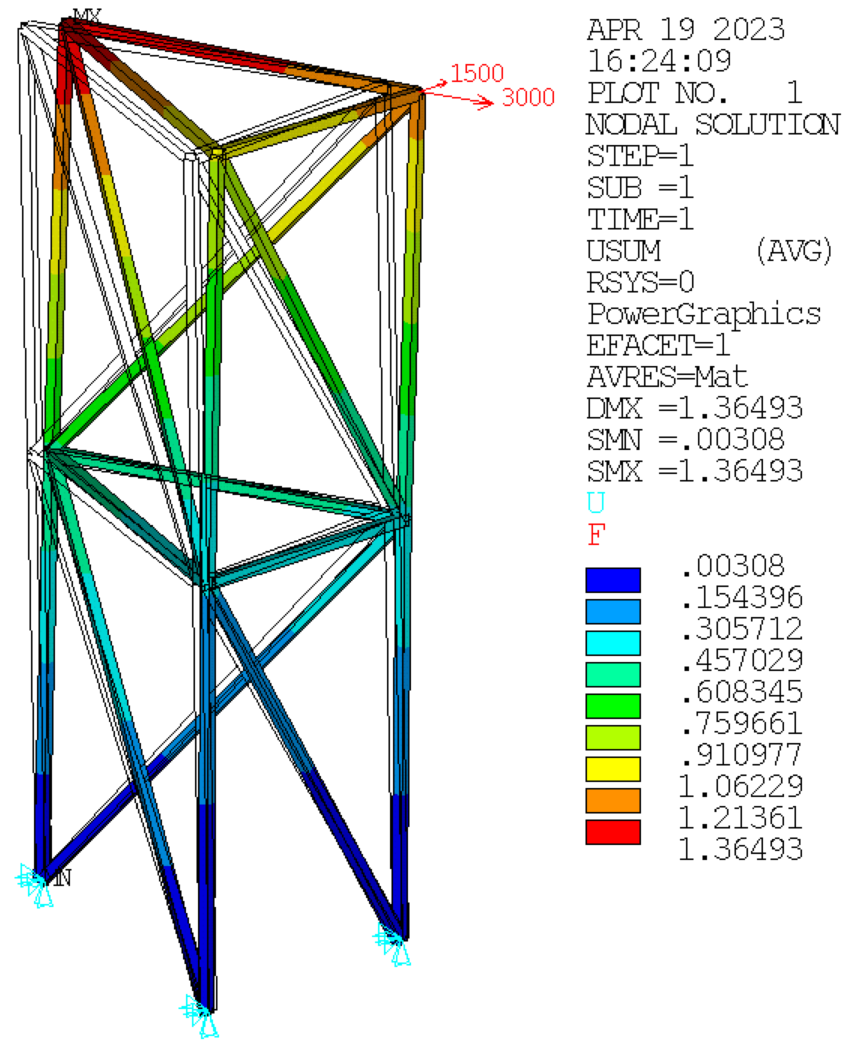




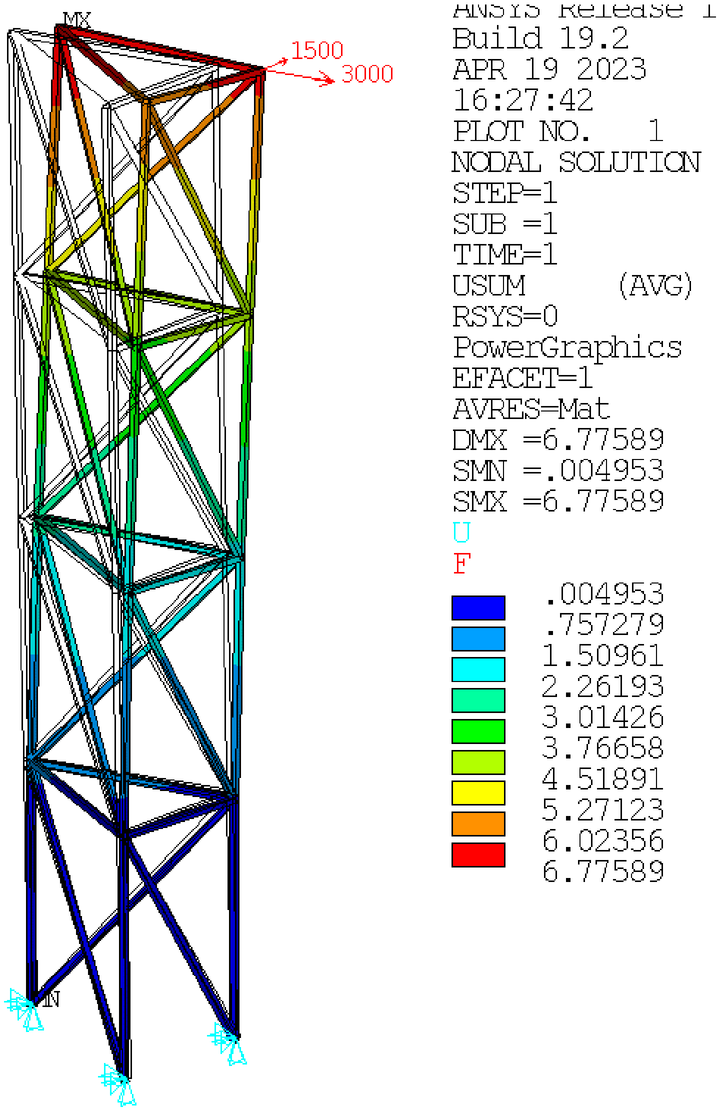


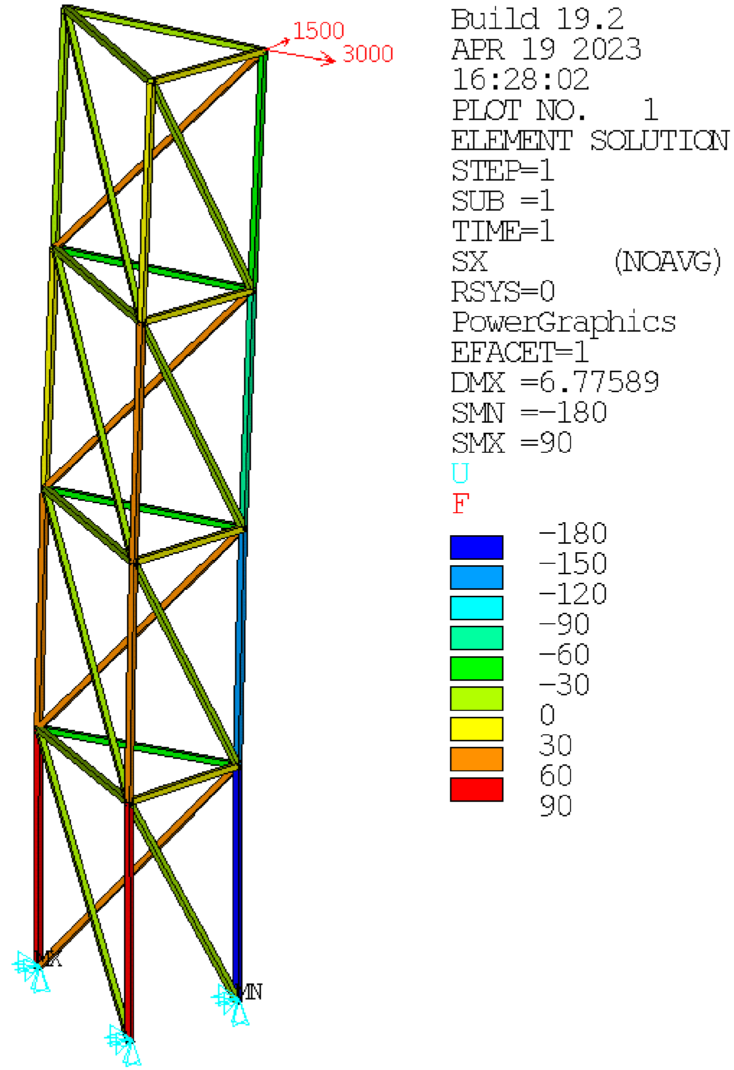
Expanding the model in ANSYS



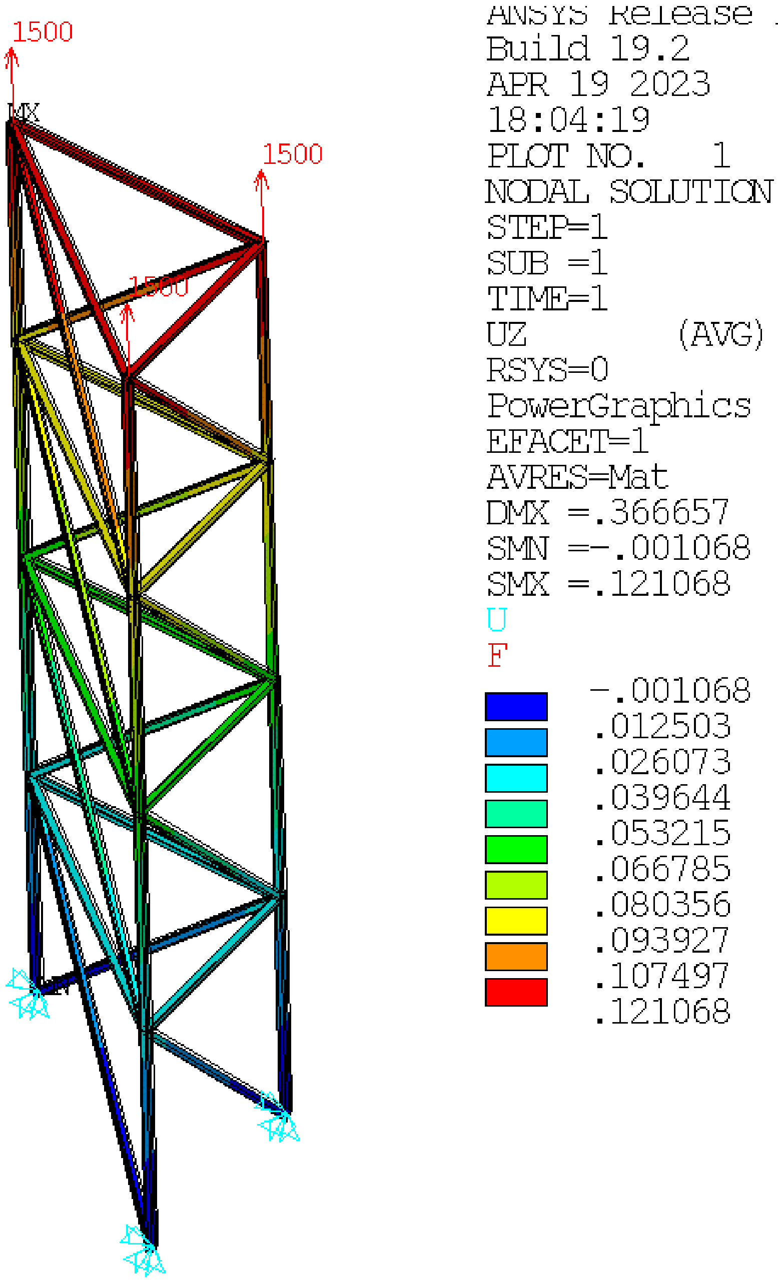


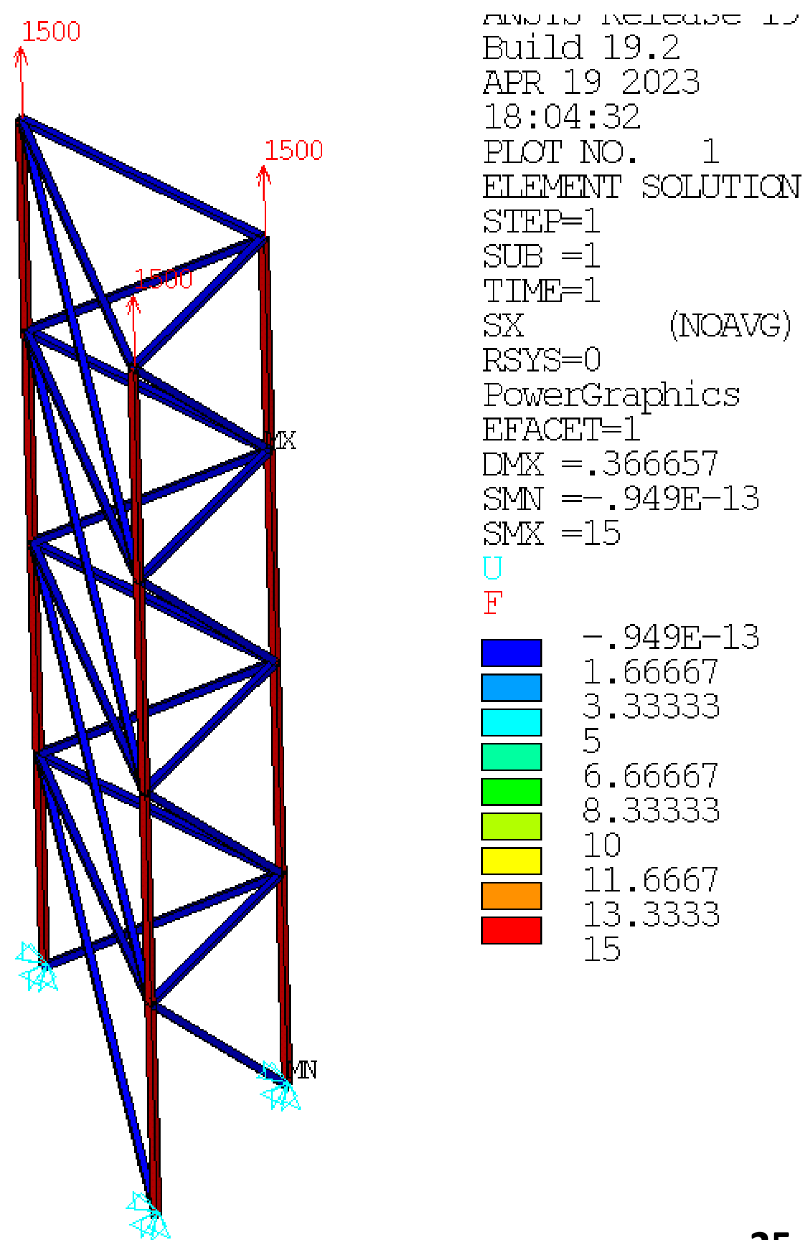
Expanding the model in ANSYS





Stretching an extended model in ANSYS





Twisting an extended model in ANSYS